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A STABILITY ANALYSIS OF THE PERFECTLY MATCHED LAYER METHOD

S. Joe Yakura, David Dietz, Andy Greenwood, and Ernest Baca

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Final Report

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Directed Energy Directorate
3550 Aberdeen Ave SE
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S. Joe Yakura

S. JOE YAKURA, DR-III, (GS-14)
Project Manager

FOR THE COMMANDER

Jorge E. Beraun

JORGE E. BERAUN, DR-IV, (GS-15)
Chief, DE Effects Research Branch

R. Earl Good

R. EARL GOOD, SES
Director, Directed Energy Directorate

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S. Joe Yakura, David Dietz, Andy Greenwood and Ernest Baca

Air Force Research Laboratory, Directed Energy Directorate,
Kirtland AFB, New Mexico 87117

Abstract

We perform a detailed stability analysis based on the unsplit-field uniaxial perfectly matched layer (PML) formulation. Our finding shows that it is essential to have transverse field gradients present at all times to stabilize PML calculations. In the absence of transverse field gradients, the PML method becomes unstable with the axial field components growing linearly in time.

I. INTRODUCTION

Despite the successful implementation of the perfectly matched layer (PML) method to absorb outgoing waves at the artificial boundaries of a bounded numerical volume, the question of the stability of the PML method remains [1,2,3]. Abarbanel and Gottlieb [1] carried out a detailed stability analysis of Berenger's split-field PML formulation [4], and they concluded that the split-field PML equations are not mathematically strongly well-posed. Hence, these equations result in unstable field components that diverge linearly in time.

In this paper we present a stability analysis starting with the unsplit-field uniaxial PML formulation [5,6], and derive a stability condition for the simple, nondispersive PML equations. The analysis shows that in rare instances the PML method results in an unstable condition. However, for PML parameter values used in most practical applications, the PML method is stable.

II. MAXWELL'S EQUATIONS INSIDE PML

For a plane wave propagating in the arbitrary x-y direction into a uniaxial anisotropic medium, the 2-D PML equations in the frequency (ω) domain are given by [5,6]

$$\underline{\nabla} \times \underline{H}(\omega; x, y) = j\omega\varepsilon_0\varepsilon_R S^{PML}(\omega; x)\,\underline{E}(\omega; x, y), \tag{II.1}$$

$$\underline{\nabla} \times \underline{E}(\omega; x, y) = -j\omega\mu_0\mu_R S^{PML}(\omega; x)\,\underline{H}(\omega; x, y), \tag{II.2}$$

where $\underline{E}(\omega;x,y)$, $\underline{H}(\omega;x,y)$, ε_0, ε_R, μ_0, μ_R and $S^{PML}(\omega)$ are the electric field vector, the magnetic field vector, the free-space permittivity, the relative permittivity, the free-space permeability, the relative permeability and the uniaxial anisotropic PML tensor, respectively. Elements of the uniaxial anisotropic PML tensor, $S^{PML}(\omega)$, are given by

$$S^{PML}(\omega) = \begin{pmatrix} \frac{1}{S_x(\omega)} & 0 & 0 \\ 0 & S_x(\omega) & 0 \\ 0 & 0 & S_x(\omega) \end{pmatrix}, \tag{II.3}$$

where $S_x(\omega)$ is an arbitrarily defined ω and x dependent function. It is a common practice in the FDTD community to choose $S_x(\omega)$ in the form

$$S_x(\omega) = 1 + \frac{\sigma_x}{j\omega\varepsilon_0\varepsilon_R} \quad \text{with} \quad \frac{\sigma_x}{\varepsilon_0\varepsilon_R} = \frac{\sigma_x^*}{\mu_0\mu_R} \tag{II.4}$$

for the PML matching condition. The quantities σ_x and σ_x^* represent electric and magnetic conductivities arbitrarily introduced in order to implement the PML method.

Since Maxwell's equations in 2-D decompose into transverse magnetic (TM_z) and transverse electric (TE_z) waves, these waves are considered separately in the stability analysis.

II.A TRANSVERSE MAGNETIC (TM_z) WAVE

For TM_z waves, Eqs. (II.1) through (II.4) reduce to the following three equations for three field components $H_x(\omega;x,y)$, $H_y(\omega;x,y)$ and $E_z(\omega;x,y)$:

$$[\underline{\nabla} \times \underline{H}(\omega;x,y)]_z = j\omega\varepsilon_0\varepsilon_R\left(1 + \frac{\sigma_x}{j\omega\varepsilon_0\varepsilon_R}\right)E_z(\omega;x,y), \tag{II.5}$$

$$[\underline{\nabla} \times \underline{E}(\omega;x,y)]_x = \frac{-j\omega\mu_0\mu_R}{\left(1 + \frac{\sigma_x}{j\omega\varepsilon_0\varepsilon_R}\right)}H_x(\omega;x,y), \tag{II.6}$$

$$[\underline{\nabla} \times \underline{E}(\omega;x,y)]_y = -j\omega\mu_0\mu_R\left(1 + \frac{\sigma_x}{j\omega\varepsilon_0\varepsilon_R}\right)H_y(\omega;x,y). \tag{II.7}$$

Taking the inverse Fourier transform of the above equations results in the following time-dependent forms:

$$\frac{\partial E_z(t;x,y)}{\partial t} + \frac{\sigma_x}{\varepsilon_0\varepsilon_R}E_z(t;x,y) + \frac{1}{\varepsilon_0\varepsilon_R}\frac{\partial H_x(t;x,y)}{\partial y} - \frac{1}{\varepsilon_0\varepsilon_R}\frac{\partial H_y(t;x,y)}{\partial x} = 0, \tag{II.8}$$

$$\frac{\partial H_x(t;x,y)}{\partial t} - \frac{\sigma_x}{\varepsilon_0\varepsilon_R}H_x(t;x,y) + \left(\frac{\sigma_x}{\varepsilon_0\varepsilon_R}\right)^2 V_x^H(t;x,y) + \frac{1}{\mu_0\mu_R}\frac{\partial E_z(t;x,y)}{\partial y} = 0, \tag{II.9}$$

$$\frac{\partial V_x^H(t;x,y)}{\partial t} + \frac{\sigma_x}{\varepsilon_0\varepsilon_R}V_x^H(t;x,y) - H_x(t;x,y) = 0, \tag{II.10}$$

$$\frac{\partial H_y(t;x,y)}{\partial t} + \frac{\sigma_x}{\varepsilon_0\varepsilon_R}H_y(t;x,y) - \frac{1}{\mu_0\mu_R}\frac{\partial E_z(t;x,y)}{\partial x} = 0. \tag{II.11}$$

In the above, the first order time-dependent equation [Eq. (II.10)] for $V_x^H(t;x,y)$ is introduced to handle the delayed time-response of $H_x(t;x,y)$. This equation follows naturally from the inverse Fourier transform of $H_x(\omega;x,y)$ when $V_x^H(t;x,y)$ is defined in the following integral form:

$$V_x^H(t;x,y) = \int_{-\infty}^{t} H_x(t';x,y)\,\exp\left[-\left(\frac{\sigma_x}{\varepsilon_0\varepsilon_R}\right)(t-t')\right]dt'. \tag{II.12}$$

Casting Eqs. (II.8) through (II.11) into a more compact form results in

$$\frac{\partial \underline{W}^{TM}}{\partial t} + \underline{\underline{A}}^{TM} \bullet \underline{W}^{TM} + \underline{\underline{B}}^{TM} \bullet \frac{\partial \underline{W}^{TM}}{\partial x} + \underline{\underline{C}}^{TM} \bullet \frac{\partial \underline{W}^{TM}}{\partial y} = 0, \tag{II.13}$$

where $\underline{W}^{TM} = \{E_z(t;x,y), H_x(t;x,y), V_x^H(t;x,y), H_y(t;x,y)\}^T$, $\bullet$ is used to denote matrix multiplication, and matrix coefficients $\underline{\underline{A}}^{TM}$, $\underline{\underline{B}}^{TM}$ and $\underline{\underline{C}}^{TM}$ are given by

$$\underline{\underline{A}}^{TM} = \begin{pmatrix} \frac{\sigma_X}{\varepsilon_0\varepsilon_R} & 0 & 0 & 0 \\ 0 & \frac{-\sigma_X}{\varepsilon_0\varepsilon_R} & \left(\frac{\sigma_X}{\varepsilon_0\varepsilon_R}\right)^2 & 0 \\ 0 & -1 & \frac{\sigma}{\varepsilon_0\varepsilon_R} & 0 \\ 0 & 0 & 0 & \frac{\sigma_X}{\varepsilon_0\varepsilon_R} \end{pmatrix}, \tag{II.14}$$

$$\underline{\underline{B}}^{TM} = \begin{pmatrix} 0 & 0 & 0 & \frac{-1}{\varepsilon_0\varepsilon_R} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{-1}{\mu_0\mu_R} & 0 & 0 & 0 \end{pmatrix}, \quad \underline{\underline{C}}^{TM} = \begin{pmatrix} 0 & \frac{1}{\varepsilon_0\varepsilon_R} & 0 & 0 \\ \frac{1}{\mu_0\mu_R} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{II.15-16}$$

To carryout the stability analysis of Eq. (II.13), the Laplace transform is performed in the time domain and the Fourier transform in the spatial domain. The Laplace and Fourier transform function $\underline{\hat{W}}^{TM}(s;k_x,k_y)$ of $\underline{W}^{TM}(t;x,y)$ is defined by

$$\underline{W}^{TM}(t;x,y) = \frac{1}{2\pi j}\int_{s_0-j\infty}^{s_0+j\infty} \exp(st)ds \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \underline{\hat{W}}^{TM}(s;k_x,k_y)\exp(jk_x x + jk_y y)dk_x dk_y \quad \text{with } \mathrm{Re}(s) \geq s_0. \tag{II.17}$$

Upon performing both Laplace and Fourier transforms, Eq. (II.13) becomes

$$\underline{\underline{\Omega}}^{TM}(s;k_x,k_y) \bullet \underline{\hat{W}}^{TM}(s;k_x,k_y) = \underline{\hat{W}}^{TM}(0;k_x,k_y), \tag{II.18}$$

where $\underline{\underline{\Omega}}^{TM}(s;k_x,k_y)$ is the characteristic matrix of the TM_z wave defined by

$$\underline{\underline{\Omega}}^{TM}(s;k_x,k_y) = \begin{pmatrix} s+\frac{\sigma_x}{\varepsilon_0\varepsilon_R} & \frac{jk_y}{\varepsilon_0\varepsilon_R} & 0 & \frac{-jk_x}{\varepsilon_0\varepsilon_R} \\ \frac{jk_y}{\mu_0\mu_R} & s-\frac{\sigma_x}{\varepsilon_0\varepsilon_R} & \left(\frac{\sigma_x}{\varepsilon_0\varepsilon_R}\right)^2 & 0 \\ 0 & -1 & s+\frac{\sigma_x}{\varepsilon_0\varepsilon_R} & 0 \\ \frac{-jk_x}{\mu_0\mu_R} & 0 & 0 & s+\frac{\sigma_x}{\varepsilon_0\varepsilon_R} \end{pmatrix}. \tag{II.19}$$

The stability of the system is characterized by investigating the determinant (or equivalently the eigenvalues) of the characteristic matrix $\underline{\underline{\Omega}}^{TM}(s;k_x,k_y)$. The determinant gives the following quartic algebraic equation:

$$s^2\left[\left(s+\frac{\sigma_x}{\varepsilon_0\varepsilon_R}\right)^2+\frac{(k_x)^2}{\varepsilon_0\varepsilon_R\mu_0\mu_R}\right]+\left(s+\frac{\sigma_x}{\varepsilon_0\varepsilon_R}\right)^2\frac{(k_y)^2}{\varepsilon_0\varepsilon_R\mu_0\mu_R}=0. \tag{II.20}$$

II.B TRANSVERSE ELECTRIC (TE_z) WAVE

For TE_z waves, Eqs. (II.1) through (II.4) reduce to the following three equations for three field components $H_z(\omega;x,y)$, $E_x(\omega;x,y)$ and $E_y(\omega;x,y)$:

$$[\underline{\nabla}\times\underline{E}(\omega;x,y)]_z=-j\omega\mu_0\mu_R\left(1+\frac{\sigma_x}{j\omega\varepsilon_0\varepsilon_R}\right)H_z(\omega;x,y), \tag{II.21}$$

$$[\underline{\nabla}\times\underline{H}(\omega;x,y)]_x=\frac{j\omega\varepsilon_0\varepsilon_R}{\left(1+\frac{\sigma_x}{j\omega\varepsilon_0\varepsilon_R}\right)}E_x(\omega;x,y), \tag{II.22}$$

$$[\underline{\nabla}\times\underline{H}(\omega;x,y)]_y=j\omega\varepsilon_0\varepsilon_R\left(1+\frac{\sigma_x}{j\omega\varepsilon_0\varepsilon_R}\right)E_y(\omega;x,y). \tag{II.23}$$

Following the same steps as in the TM_z wave case yields the following equations for the TE_z wave in the Laplace and Fourier domains:

$$\underline{\underline{\Omega}}^{TE}(s;k_x,k_y)\bullet\underline{\hat{W}}^{TE}(s;k_x,k_y)=\underline{\hat{W}}^{TE}(0;k_x,k_y), \tag{II.24}$$

where $\underline{\hat{W}}^{TE}(s;k_x,k_y)=\{H_z(s;k_x,k_y), E_x(s;k_x,k_y), V_x^E(s;k_x,k_y), E_y(s;k_x,k_y)\}^T$, and $\underline{\underline{\Omega}}^{TE}(s;k_x,k_y)$ is the characteristic matrix of the TE_z wave defined by

$$\underline{\underline{\Omega}}^{TE}(s;k_x,k_y)=\begin{pmatrix} s+\frac{\sigma_x}{\varepsilon_0\varepsilon_R} & \frac{-jk_y}{\mu_0\mu_R} & 0 & \frac{jk_x}{\mu_0\mu_R} \\ \frac{-jk_y}{\varepsilon_0\varepsilon_R} & s-\frac{\sigma_x}{\varepsilon_0\varepsilon_R} & \left(\frac{\sigma_x}{\varepsilon_0\varepsilon_R}\right)^2 & 0 \\ 0 & -1 & s+\frac{\sigma_x}{\varepsilon_0\varepsilon_R} & 0 \\ \frac{jk_x}{\varepsilon_0\varepsilon_R} & 0 & 0 & s+\frac{\sigma_x}{\varepsilon_0\varepsilon_R} \end{pmatrix}. \tag{II.25}$$

Taking the determinant of the characteristic matrix, $\underline{\underline{\Omega}}^{TE}(s;k_x,k_y)$, gives the following characteristic equation which is exactly the same as in the TM_z wave case:

$$s^2\left[\left(s+\frac{\sigma_x}{\varepsilon_0\varepsilon_R}\right)^2+\frac{(k_x)^2}{\varepsilon_0\varepsilon_R\mu_0\mu_R}\right]+\left(s+\frac{\sigma_x}{\varepsilon_0\varepsilon_R}\right)^2\frac{(k_y)^2}{\varepsilon_0\varepsilon_R\mu_0\mu_R}=0. \tag{II.26}$$

III. STABILITY ANALYSIS FOR A UNIAXIAL PML MEDIUM

To study Eq. (II.20) [or Eq. (II.26)] we first normalize s, k_x and k_y by setting

$$\frac{s}{\left(\frac{\sigma_x}{\varepsilon_0\varepsilon_R}\right)} \to S,\quad \frac{\left(\frac{(k_x)^2}{\varepsilon_0\varepsilon_R\mu_0\mu_R}\right)}{\left(\frac{\sigma_x}{\varepsilon_0\varepsilon_R}\right)^2} \to (K_x)^2 \quad \text{and} \quad \frac{\left(\frac{(k_y)^2}{\varepsilon_0\varepsilon_R\mu_0\mu_R}\right)}{\left(\frac{\sigma_x}{\varepsilon_0\varepsilon_R}\right)^2} \to (K_y)^2. \qquad \text{(III.1-3)}$$

Now, Eq. (II.20) [or Eq. (II.26)] results in the following form:

$$S^2[(S+1)^2+(K_x)^2]+(S+1)^2(K_y)^2=0. \qquad \text{(III.4)}$$

From this expression it is immediately apparent that if K_y is zero then two of the four roots are located at $S = 0$ in the complex S-plane, which results in unconditionally unstable behavior that grows linearly with time. These two roots are associated with the axial field component and the delayed time-response function of the axial field component. The other two roots are related to the incoming and outgoing damped transverse waves that propagate as $\exp[-(\sigma_x/\varepsilon_0\varepsilon_R)\, t \pm jk_x x]$.

On the other hand, if both K_x and K_y are real numbers then Eq. (III.4) has four complex roots [i.e. two sets of complex conjugate roots], and that the real parts of these roots can be shown to be all negative [see Appendix A]. Thus, all eigenfunctions associated with these eigenvalues are well-behaved and stable.

As seen in Eq. (III.4), the term that contributes to stabilizing the system is the real part of K_y. Physically, this means that the transverse field gradients (in the y-direction for the present analysis) contribute to stabilize axial field (in the x-direction for the present analysis) components as TM_z and TE_z waves propagate into a uniaxial anisotropic PML medium.

Unfortunately, the actual PML system is not typically characterized by real K_x but rather by complex K_x because of the evanescent behavior of the propagating wave into the uniaxial PML medium. To investigate the effect of the imaginary part of K_x on the stability of a system, we solve Eq. (III.4) directly using MATHEMATICA software [7]. The exact expressions for the four complex roots are shown in Appendix B. Calculations show that for $\mathrm{Im}\{|K_x|\} \gg \mathrm{Re}\{|K_x|\}$ it is possible for the real parts of the roots to become positive. However, in usual implementation of the PML method $\mathrm{Im}\{|K_x|\} \gg \mathrm{Re}\{|K_x|\}$ is not normally satisfied; thus, it is unlikely that PML calculations become unstable for practical PML applications.

IV. EXTENDING THE STABILITY ANALYSIS TO CORNER REGIONS

At a corner region of the 2D PML medium, the uniaxial anisotropic PML tensor, $S^{PML}(\omega)$, has to be modified to include contributions in both x and y directions as follows:

$$S^{PML}(\omega) = \begin{pmatrix} \frac{S_y(\omega)}{S_x(\omega)} & 0 & 0 \\ 0 & \frac{S_x(\omega)}{S_y(\omega)} & 0 \\ 0 & 0 & S_x(\omega)S_y(\omega) \end{pmatrix}, \qquad \text{(IV.1)}$$

where $S_y(\omega)$ is defined in the same way as $S_x(\omega)$ to take the form

$$S_y(\omega) = 1+\frac{\sigma_y}{j\omega\varepsilon_0\varepsilon_R} \quad \text{with} \quad \frac{\sigma_y}{\varepsilon_0\varepsilon_R} = \frac{\sigma_y^*}{\mu_0\mu_R}. \qquad \text{(IV.2)}$$

Using Eqs. (IV.1) in Eqs. (II.1) and (II.2), and following the same steps as in previous sections yields the following equations for the TM_z wave in the Laplace and Fourier domains at the 2D PML corner region:

$$\left(\underline{\underline{\Omega}}^{TM}(s;k_x,k_y)\right)_{corner} \bullet \left(\underline{\hat{W}}^{TM}(s;k_x,k_y)\right)_{corner} = \left(\underline{\hat{W}}^{TM}(0;k_x,k_y)\right)_{corner}, \qquad \text{(IV.3)}$$

where $(\underline{\hat{W}}^{TM}(s;k_x,k_y))_{corner} = \{E_z(s;k_x,k_y), H_x(s;k_x,k_y), V_x^H(s;k_x,k_y), H_y(s;k_x,k_y), V_y^H(s;k_x,k_y)\}^T$, and $(\underline{\underline{\Omega}}^{TM}(s;k_x,k_y))_{corner}$ is the characteristic matrix of the TM_z defined by

$$(\underline{\underline{\Omega}}^{TM}(s;k_x,k_y))_{corner} = \begin{pmatrix} \left(s+\frac{\sigma_x+\sigma_y}{\varepsilon_0\varepsilon_R}+\frac{\sigma_x\sigma_y}{s\,(\varepsilon_0\varepsilon_R)^2}\right) & \frac{jk_y}{\varepsilon_0\varepsilon_R} & 0 & \frac{-jk_x}{\varepsilon_0\varepsilon_R} & 0 \\ \frac{jk_y}{\mu_0\mu_R} & s-\frac{\sigma_x-\sigma_y}{\varepsilon_0\varepsilon_R} & \left(\frac{\sigma_x}{\varepsilon_0\varepsilon_R}\left(\frac{\sigma_x-\sigma_y}{\varepsilon_0\varepsilon_R}\right)\right) & 0 & 0 \\ 0 & -1 & s+\frac{\sigma_x}{\varepsilon_0\varepsilon_R} & 0 & 0 \\ \frac{-jk_x}{\mu_0\mu_R} & 0 & 0 & s-\frac{\sigma_y-\sigma_x}{\varepsilon_0\varepsilon_R} & \left(\frac{\sigma_y}{\varepsilon_0\varepsilon_R}\left(\frac{\sigma_y-\sigma_x}{\varepsilon_0\varepsilon_R}\right)\right) \\ 0 & 0 & 0 & -1 & s+\frac{\sigma_y}{\varepsilon_0\varepsilon_R} \end{pmatrix} \qquad \text{(IV.4)}$$

Taking the determinant of the characteristic matrix, $(\underline{\underline{\Omega}}^{TM}(s;k_x,k_y))_{corner}$, gives the following characteristic equation:

$$s\left[\left(s+\frac{\sigma_x}{\varepsilon_0\varepsilon_R}\right)^2\left(s+\frac{\sigma_y}{\varepsilon_0\varepsilon_R}\right)^2+\left(s+\frac{\sigma_y}{\varepsilon_0\varepsilon_R}\right)^2\frac{(k_x)^2}{\varepsilon_0\varepsilon_R\mu_0\mu_R}+\left(s+\frac{\sigma_x}{\varepsilon_0\varepsilon_R}\right)^2\frac{(k_y)^2}{\varepsilon_0\varepsilon_R\mu_0\mu_R}\right]=0. \qquad \text{(IV.5)}$$

As seen in the above equation one root is located at $s = 0$, which gives a stable solution in the time domain, and the other four roots can be obtained by setting the expression inside the square bracket to zero. For real values of k_x and k_y, a procedure similar to that in Appendix A shows the real parts of the four complex roots are always negative, implying stable solutions in the time domain. For arbitrary complex values of k_x and k_y, the real parts of the four complex roots have to be investigated numerically from the exact expressions shown in Appendix B.

For the special case of $\sigma_x = \sigma_y$ the equation formed by setting the expression inside the square bracket to zero can be solved exactly to obtain an analytical expression for the stability condition; in this case the square bracket term reduces to

$$\left(s+\frac{\sigma_x}{\varepsilon_0\varepsilon_R}\right)^2\left[\left(s+\frac{\sigma_x}{\varepsilon_0\varepsilon_R}\right)^2+\frac{(k_x)^2+(k_x)^2}{\varepsilon_0\varepsilon_R\mu_0\mu_R}\right]=0. \qquad \text{(IV.6)}$$

Solving Eq. (IV.6) results in the double root $s = -(\sigma_x/\varepsilon_0\varepsilon_R)$, which give stable solutions in the time domain, and the other two roots $s = -(\sigma_x/\varepsilon_0\varepsilon_R) \pm j\{\sqrt{1/(\varepsilon_0\varepsilon_R\mu_0\mu_R)}\; k\}$ where $(k)^2 = (k_x)^2 + (k_y)^2$. Expressing k in terms of its real and imaginary parts as $k = k^R + jk^I$, the two roots $s = -(\sigma_x/\varepsilon_0\varepsilon_R) \pm j\{\sqrt{1/(\varepsilon_0\varepsilon_R\mu_0\mu_R)}\,(k^R + jk^I)\}$ can be expressed as

$$s = -\frac{\sigma_x}{\varepsilon_0\varepsilon_R} \pm j\left\{\sqrt{\frac{[(k^R)^2+(k^I)^2]}{\varepsilon_0\varepsilon_R\mu_0\mu_R}}\exp\left[j\tan^{-1}\left(\frac{k^I}{k^R}\right)\right]\right\}, \quad (k^R \neq 0). \qquad \text{(IV.7)}$$

In the above expression, one of the two roots gives a positive real value if the following condition is satisfied:

$$\sqrt{\frac{[(k^R)^2+(k^I)^2]}{\varepsilon_0\varepsilon_R\mu_0\mu_R}}\sin\left[\tan^{-1}\left(\frac{k^I}{k^R}\right)\right] > \frac{\sigma_x}{\varepsilon_0\varepsilon_R} \tag{IV.8}$$

or

$$k^I > \sqrt{\frac{\mu_0\mu_R}{\varepsilon_0\varepsilon_R}}\,\sigma_x . \tag{IV.9}$$

Hence, the PML system becomes unstable if the above condition is met for the case $\sigma_x = \sigma_y$.

The stability analysis of TE_z waves in corner regions results in the same stability condition as for TM_z waves.

V. CONCLUSIONS

Starting with unsplit-field uniaxial PML formulation in the frequency domain, Maxwell's equations are cast into a set of first order differential equations in time. Then, using the Laplace and Fourier transforms, the characteristic equation of a system is obtained and investigated for its dynamic stability.

From stability analysis, we find that it is essential for the transverse field gradients to be present at all times in order to stabilize PML calculations. In fact, in the absence of transverse field gradients the PML method becomes unstable with the axial field components growing linearly in time.

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Appendix A

A proof to show that the real parts of the roots of the polynomial, $S^2[(1+S)^2+a]+(1+S)^2 b=0$ with positive real coefficients a and b, are all negative.

Theorem:

Consider the equation

$$S^2[(S+1)^2+a]+(S+1)^2 b=0,\quad S\in\mathbb{C} \tag{A.1}$$

where a, b > 0. Then

(i) Eq. (A.1) has no real solutions;

(ii) If $S_i = \alpha_i + j\beta_i$, i =1, 2, 3, 4 ($\beta_i \neq 0$) are the roots of Eq. (A.1) then $\alpha_i < 0$, i = 1, 2, 3, 4.

Proof:

(i) Rewrite Eq. (A.1) as

$$S^2[(S+1)^2 + a] = -(S+1)^2 b, \tag{A.2}$$

and note that, since a, b > 0, if $S \in \mathbb{R} \setminus \{-1,0\}$ then the LHS of Eq. (A.2) > 0 and the RHS of Eq. (A.2) < 0, a contradiction; while if S = -1 then the LHS of Eq. (A.2) > 0 and the RHS of Eq. (A.2) = 0 and if S = 0 then the LHS of Eq. (A.2) = 0 and the RHS of Eq. (A.2) < 0, also contradictions. Thus, if S satisfies Eq. (A.1) then $S \notin \mathbb{R}$. Further, all four solutions are of the form $S = \alpha + j\beta$, $\alpha, \beta \in \mathbb{R}$, $\beta \neq 0$.

(ii) If $S = \alpha + j\beta$ is any solution of Eq. (A.1) then

$$(\alpha + j\beta)^2[(\alpha + j\beta + 1)^2 + a] + (\alpha + j\beta + 1)^2 b = 0. \tag{A.3}$$

Expanding and equating real and imaginary parts of Eq. (A.3) to zero gives

$$(\alpha^2 - \beta^2)^2 - 4\alpha^2\beta^2 + 2[\alpha(\alpha^2 - \beta^2) - 2\alpha\beta^2] + (1 + a + b)(\alpha^2 - \beta^2) + 2\alpha b + b = 0 \tag{A.4}$$

and

$$\beta[2\alpha(\alpha^2 - \beta^2) + 2\alpha^2 + (\alpha^2 - \beta^2) + (1 + a + b)\alpha + b] = 0. \tag{A.5}$$

In Eq. (A.5) the term in brackets must be equal to zero since, otherwise, $\beta = 0$, which is not possible since $S \notin \mathbb{R}$. Rewritten the term results in

$$-(2\alpha + 1)\beta^2 + 2\alpha^3 + 3\alpha^2 + (1 + a + b)\alpha + b = 0. \tag{A.6}$$

If $\alpha = -½$ then we are done (since then $\alpha < 0$). Otherwise, if $\alpha \neq -½$ then Eq. (A.6) gives

$$\beta^2 = \frac{1}{2\alpha + 1}[2\alpha^3 + 3\alpha^2 + (1 + a + b)\alpha + b] \tag{A.7}$$

and

$$\beta^2 - \alpha^2 = \frac{1}{2\alpha + 1}[2\alpha^2 + (1 + a + b)\alpha + b]. \tag{A.8}$$

Substituting Eqs. (A.7) and (A.8) into Eq. (A.4) and simplifying leads to

$$\begin{aligned} &16\alpha^6 + 48\alpha^5 + 8(7 + a + b)\alpha^4 + 16(2 + a + b)\alpha^3 \\ &\quad + [(1 + a + b)^2 + 8(1 + a + b)]\alpha^2 + (1 + a + b)^2\alpha + ab = 0. \end{aligned} \tag{A.9}$$

Since a, b > 0 then all coefficients in Eq. (A.9) are > 0; thus, by Descartes's Rule of Signs [8], Eq. (A.9) has no positive roots. Further, zero is not a root of Eq. (A.9). Hence, all real roots of Eq. (A.9) are < 0. Finally, α_i, i = 1, 2, 3, 4 must be among the solutions of Eq. (A.9) so $\alpha_i < 0$, i = 1, 2, 3, 4.

Appendix B

Four complex roots of the polynomial, $S^4 + a\,S^3 + b\,S^2 + c\,S + d = 0$, with complex coefficients a, b, c and d are given by [7]

$$S_1 = -\frac{a}{4} - \frac{1}{2}\sqrt{\frac{a^2}{4} - \frac{2b}{3} + \Pi} - \frac{1}{2}\sqrt{\frac{a^2}{2} - \frac{4b}{3} - \Pi - \frac{(-a^3 + 4ab - 8c)}{4\left(\sqrt{\frac{a^2}{4} - \frac{2b}{3} + \Pi}\right)}}, \tag{B.1}$$

$$S_2 = -\frac{a}{4} - \frac{1}{2}\sqrt{\frac{a^2}{4} - \frac{2b}{3} + \Pi} + \frac{1}{2}\sqrt{\frac{a^2}{2} - \frac{4b}{3} - \Pi - \frac{(-a^3 + 4ab - 8c)}{4\left(\sqrt{\frac{a^2}{4} - \frac{2b}{3} + \Pi}\right)}}, \tag{B.2}$$

$$S_3 = -\frac{a}{4} + \frac{1}{2}\sqrt{\frac{a^2}{4} - \frac{2b}{3} + \Pi} - \frac{1}{2}\sqrt{\frac{a^2}{2} - \frac{4b}{3} - \Pi + \frac{(-a^3 + 4ab - 8c)}{4\left(\sqrt{\frac{a^2}{4} - \frac{2b}{3} + \Pi}\right)}}, \tag{B.3}$$

$$S_4 = -\frac{a}{4} + \frac{1}{2}\sqrt{\frac{a^2}{4} - \frac{2b}{3} + \Pi} + \frac{1}{2}\sqrt{\frac{a^2}{2} - \frac{4b}{3} - \Pi + \frac{(-a^3 + 4ab - 8c)}{4\left(\sqrt{\frac{a^2}{4} - \frac{2b}{3} + \Pi}\right)}}, \tag{B.4}$$

where

$$\Pi = \frac{(2^{1/3})\Phi}{3\Psi} + \frac{\Psi}{3(2^{1/3})}, \tag{B.5}$$

$$\Phi = b^2 - 3ac + 12d, \tag{B.6}$$

$$\Psi = \left(\Gamma + \sqrt{-4\Phi^3 + \Gamma^2}\right)^{1/3}, \tag{B.7}$$

$$\Gamma = 2b^3 - 9abc + 27c^2 + 27a^2d - 72bd. \tag{B.8}$$

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